Midterm assignment (due 27/11/2019)

S. Allais, M. Joseph

Every result seen during tutorials can be used without proof.

We admit that the Lie Bracket $[X,Y](p)$ of two vector fields X, Y computed at $p \in M$ is equal to $c'(0)$ where $c :]-\varepsilon, \varepsilon[\to M$ is the differentiable path $c(t) = Y^{-\sqrt{t}} \circ X^{-\sqrt{t}} \circ Y^{\sqrt{t}} \circ X^{\sqrt{t}}(p)$ and $\varepsilon > 0$ is small enough.

Exercise 1 (Exponential map of a Lie group)**.** Let G be a Lie group and $e \in G$ be its neutral element. For all $g \in G$, let $L_g : G \to G$ be the left multiplication by g. A vector field $X \in \mathcal{L}(G)$ is said to be *left-invariant* if for all g in G, $(L_g)_* X = X$. Let $\mathcal{L}(G)$ be the set of left-invariant vector field on G. We recall that there is a natural isomorphism of vector spaces $T_eG \to \mathcal{L}(G)$ given by $x \mapsto V_x$ where

$$\forall g \in G, \quad V_x(g) := \mathrm{d}(L_g)_e \cdot x.$$

1. Given $x \in T_eG$, let Φ_x^t be the flow of the vector field V_x at time t (a priori not defined on the whole manifold G).
   (a) Show that the flow operator is left-invariant: for all $g, h \in G$, if $\Phi_x^t(h)$ is well defined then so is $\Phi_x^t(gh)$ and $\Phi_x^t(gh) = g\Phi_x^t(h)$.
   (b) Show that if $\Phi_x^t(e)$ and $\Phi_x^s(e)$ are well defined then so is $\Phi_x^{t+s}(e)$ and $\Phi_x^{t+s}(e) = \Phi_x^t(e)\Phi_x^s(e)$. Deduce that V_x is a complete vector field.
2. Let $\exp : T_eG \to G$ be the map

$$\forall x \in T_eG, \quad \exp(x) = \Phi_x^1(e).$$

   By general results about the dependence of a solution of a differential equation on initial condition, exp is smooth and there exists some open neighborhood $U \subset T_eG$ of 0 and $V \subset G$ of e such that $\exp|_U : U \to V$ is a diffeomorphism.

   (a) Show that

$$\forall x \in T_eG, \forall t \in \mathbb{R}, \quad \exp(tx) = \Phi_x^t(e),$$

   deduce that $\exp((t+s)x) = \exp(tx)\exp(sx)$, $\frac{\mathrm{d}}{\mathrm{d}\,t}\exp(tx)|_{t=0} = x$ and $\exp(-x) = \exp(x)^{-1}$.

   (b) Let $W \subset U$ be an open neighborhood of 0 such that $\forall x, y \in W$, $\exp(x)\exp(y) \in V$. Let $\mu : W \times W \to U$ be the map satisfying

$$\forall x, y \in W, \quad \exp(x)\exp(y) = \exp(\mu(x,y)).$$

   Show that μ is a well defined smooth map such that

$$\forall x, y \in W, \quad \mu(x,y) = x + y + \frac{1}{2}\lambda(x,y) + o(\|x\|^2 + \|y\|^2)$$

   where $\lambda : T_eG \times T_eG \to T_eG$ is a bilinear skew-symmetric map (that is $\lambda(x,y) = -\lambda(y,x)$ or equivalently $\lambda(x,x) = 0$).

   (c) Show that for $x, y \in T_eG$ close enough to 0, one has

$$\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp(\lambda(x,y) + o(\|x\|^2 + \|y\|^2)),$$

   deduce that $\lambda(x,y) = [V_x, V_y]$ for all $x, y \in T_eG$.

3. Let G be a connected Lie group, show that G is commutative if and only if $[X, Y] = 0$ for all $X, Y \in \mathcal{L}(G)$.
   Start by proving that $\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y)$ *then prove that a continuous isomorphism of Lie groups* $f : G_1 \to G_2$ *which is a local homeomorphism in the neighborhood of the neutral element of* G_1 *is a surjection if* G_2 *is connected.*

Exercise 2.

1. Let $F : M \to N$ be a smooth map between two manifolds of the same dimension, with M compact. Let R be the set of regular values of F. Prove that the map $r \mapsto \#F^{-1}(\{r\})$ is locally constant.

2. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial map. Let $\varphi_N : \mathbb{S}^2 \to \mathbb{R}^2 \setminus \{0\}$ be the stereographic map from the north pole N of the sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, prove that the map $F = \varphi_N^{-1} \circ P \circ \varphi_N$ extends to a smooth map on $\mathbb{S}^2$.

3. Prove the fundamental theorem of algebra, i.e. prove that P admits a zero.

Exercise 3. A classical theorem of differential topology asserts that for all compact and connected manifold M, there exists a smooth function $f : M \to \mathbb{R}$ with a finite number of critical points. Moreover, by compactness any such function admit at least two critical points: a maximum and a minimum. Given $x \in M$, does there exists a smooth function $f = M \setminus \{x\} \to \mathbb{R}$ with no critical point?